On generalised Ackermann encodings – the basis issue

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Introduction

Why multisets?

- They find application in computer sciences as a generalisation of lists.
- Hereditarily finite multisets are used to write a simpler termination function w.r.t. its equivalent which uses hereditarily finite sets only.

Their encoding could help to develop a unique formalisation of the theory.

Why hypersets?

- They admit cicles in the membership relation (non-wellfoundedness); bisimilarity is assumed as equality criterion.
- Hereditarily finite hypersets can represent finite state automata or, more generally, graphs labelled on edges.

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Hereditarily finite sets

Definition (Hereditarily finite sets)

$$\mathsf{HF}_n = \begin{cases} \emptyset & \text{if } n = 0 \\ \mathscr{P}(\mathsf{HF}_{n-1}) & \text{if } n \in \mathbb{N}^+, \end{cases} \qquad \mathsf{HF} = \bigcup_{n\in\mathbb{N}} \mathsf{HF}_n$$

defines the cumulative hierarchy of the *hereditarily finite sets*.
Given $h \in \mathsf{HF}$, its *rank* $\mathrm{rk}(h)$ is defined as the least integer r such that $h \in \mathsf{HF}_{r+1}$.

Example

$$\emptyset \quad \{\emptyset\} \quad \{\{\emptyset\}\} \quad \{\{\emptyset\},\emptyset\} \quad \{\{\{\emptyset\}\}\} \quad \{\{\{\emptyset\}\},\emptyset\} \quad \{\{\{\emptyset\}\},\{\emptyset\}\} \quad \ldots$$

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Multisets

Definition (Multisets)

Let $O_1, \ldots, O_n$ be n distinct objects and let $m_1, \ldots, m_n \in \mathbb{N}^+$ be positive integers; then the list

$$M = [\underbrace{O_1, \ldots, O_1}_{m_1}, \ldots, \underbrace{O_n, \ldots, O_n}_{m_n}],$$

or equivalently

$$M = \{{}^{m_1}O_1, \ldots, {}^{m_n}O_n\},$$

up to any permutation, defines the *multiset* M containing the objects $O_1, \ldots, O_n$ with *multiplicities* $m_1, \ldots, m_n$ respectively. The *multiplicity map* of M and its *multiset membership relation* are then defined as

$$\mu_M(O_i) = m_i \quad \Longleftrightarrow \quad O_i \; {}^{m_i}\!\in M.$$

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Given a multiset X, define

$$\mathscr{P}^{\mu}(X) = \big\{ \{ {}^{m_1}x_1, \ldots, {}^{m_n}x_n \} \mid x_1, \ldots, x_n \in X \wedge (\forall i \neq j)(x_i \neq x_j) \wedge m_1, \ldots, m_n \in \mathbb{N}^+ \wedge n \in \mathbb{N} \big\}.$$

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Bisimilarity

Definition (Bisimulation)

A dyadic relation $\flat$ on the finite set V of the nodes of a directed graph $\mathcal{M} = (V, E)$ is said to be a *bisimulation* on $\mathcal{M}$ if $u_0 \flat u_1$ always implies that

- for every child v_1 of u_1, u_0 has at least one child v_0 s.t. $v_0 \flat v_1$, and
- for every child v_0 of u_0, u_1 has at least one child v_1 s.t. $v_0 \flat v_1$.

The largest of all bisimulations on $\mathcal{M}$ (relative to inclusion) is the following equivalence relation.

Definition (Bisimilarity)

The *bisimilarity* of a digraph $\mathcal{M}$ whose set V of nodes is finite is the dyadic relation $\equiv_{\mathcal{M}}$ over V such that $u \equiv_{\mathcal{M}} v$ holds between u, v in V if and only if $u \flat v$ holds for some bisimulation $\flat$ on $\mathcal{M}$.

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ς_0 is said to be an *hereditarily finite rational hyperset* if it can be described by a finite *set system* whose transitive closure is still finite:

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with $\varsigma_{i,j} \in \{\varsigma_0, \varsigma_1, \ldots, \varsigma_n\}$. H.f. rational hypersets are denoted by $\mathsf{HF}^{1/2}$.

Example

The hyperset $\Omega = \{\Omega\} = \{\{\Omega\}\} = \{\{\{\cdots\}\}\}$ is the one solving the set-theoretic equation $\varsigma = \{\varsigma\}$.

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The Ackermann encoding of HF

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$$\mathbb{N}_A(h) \stackrel{\text{def}}{=} \sum_{h' \in h} 2^{\mathbb{N}_A(h')} \qquad \text{for } h \in \mathsf{HF}$$

defines the *Ackermann encoding* of hereditarily finite sets.

Remark

- $\mathbb{N}_A$ is a bijection between HF and $\mathbb{N}$.
- $\mathbb{N}_A$ gives a natural, total ordering to HF.
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The encoding map $\mathbb{R}_A$

Definition ($\mathbb{R}_A^\mu$ over $\mathsf{HF}^{1/2} \cup \mathsf{HF}^\mu$)

$$\mathbb{R}_A^\mu(h) \stackrel{\text{def}}{=} \sum_{h' \in h} 2^{-\mathbb{R}_A(h')} \cdot \mu_h(h') \qquad \text{for } h \in \mathsf{HF}^{1/2} \cup \mathsf{HF}^\mu$$

defines the $\mathbb{R}_A$-*codes* of the hereditarily finite hypersets and multisets.

Example

$\mathbb{R}_A^\mu$ is not injective over HF^μ: $\mathbb{R}_A^\mu([[\emptyset],[\emptyset]]) = 1.$

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defines the $\mathfrak{A}_\beta$-*codes* of the hereditarily finite (hyper-, multi-) sets.

Remark

- $\mathfrak{A}_\beta$ has as special cases both $\mathbb{N}_A$ ($\beta = 2$) and $\mathbb{R}^\mu_A$ ($\beta = 1/2$).
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$$\mathfrak{A}_\beta(h) \quad \stackrel{\text{def}}{=} \quad \sum_{h' \in h} \mu_h(h') \cdot \beta^{\mathfrak{A}_\beta(h')} \qquad \text{for } h \in \mathsf{HF}^{1/2} \cup \mathsf{HF}^{\mu}$$

defines the $\mathfrak{A}_\beta$-*codes* of the hereditarily finite (hyper-, multi-) sets.

Remark

- $\mathfrak{A}_\beta$ has as special cases both $\mathbb{N}_A$ $(\beta = 2)$ and $\mathbb{R}^\mu_A$ $(\beta = 1/2)$.
- Whatever β is chosen, $\mathfrak{A}_\beta(\emptyset) = 0$, $\mathfrak{A}_\beta(\{\emptyset\}) = 1$.

Hereditarily finite m-multisets

Definition (m-powerset)

Given a multiset X and $m \in \mathbb{N}^+$, define

$$\mathscr{P}^{(m)}(X) = \big\{\{^{m_1}x_1, \ldots, {}^{m_n}x_n\} \mid x_1, \ldots, x_n \in X \wedge (\forall i \neq j)(x_i \neq x_j) \\ \wedge\, m_1, \ldots, m_n \in \mathbb{N}^+ \wedge (\forall i)(m_i \leq m) \wedge n \in \mathbb{N}\big\}.$$

Definition (Hereditarily finite m-multisets)

Let $m \in \mathbb{N}^+ \setminus \{1\}$; then the following defines the family of *h.f. m-multisets*.

$$\mathsf{HF}_n^{(m)} = \begin{cases} \emptyset & \text{if } n = 0 \\ \mathscr{P}^{(m-1)}(\mathsf{HF}_{n-1}^{(m)}) & \text{if } n \in \mathbb{N}^+, \end{cases} \qquad \mathsf{HF}^{(m)} = \bigcup_{n \in \mathbb{N}} \mathsf{HF}_n^{(m)}.$$

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The $\mathfrak{A}_m$ encoding of $\mathsf{HF}^{(m)}$

Theorem

Let $m \in \mathbb{N}^+ \setminus \{1\}$. Then the encoding map

$$\mathfrak{A}_m\big|_{\mathsf{HF}^{(m)}} : \quad \mathsf{HF}^{(m)} \longrightarrow \mathbb{N}$$

is bijective.

Remark

- $\mathfrak{A}_m$ is a bijection between $\mathsf{HF}^{(m)}$ and $\mathbb{N}$.
- $\mathfrak{A}_m$ gives a natural, total ordering to $\mathsf{HF}^{(m)}$.
- $H' \overset{k}{\in} H$ for $H, H' \in \mathsf{HF}^{(m)}$ if and only if there is a 'k' at position $\mathfrak{A}_m(H')$ of the m-ary expansion of $\mathfrak{A}_m(H)$.

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An example with $\beta = 3$

$\mathfrak{A}_3(h)$	$\left(\mathfrak{A}_3(h)\right)_3$	$\sum_{h' \in h} \mu_h(h') \cdot 3^{\mathfrak{A}_3(h')}$	Multiset	Corr. set
0	0	0	$\emptyset$	$\emptyset$
1	1	3^0	$[\emptyset]$	$\{\emptyset\}$
2	2	$2 \cdot 3^0$	$[\emptyset, \emptyset]$	$\{\emptyset\}$
3	10	3^1	$[[\emptyset]]$	$\{\{\emptyset\}\}$
4	11	$3^1 + 3^0$	$[[\emptyset], \emptyset]$	$\{\{\emptyset\}, \emptyset\}$
5	12	$3^1 + 2 \cdot 3^0$	$[[\emptyset], \emptyset, \emptyset]$	$\{\{\emptyset\}, \emptyset\}$
6	20	$2 \cdot 3^1$	$[[\emptyset], [\emptyset]]$	$\{\{\emptyset\}\}$
7	21	$2 \cdot 3^1 + 3^0$	$[[\emptyset], [\emptyset], \emptyset]$	$\{\{\emptyset\}, \emptyset\}$
8	22	$2 \cdot 3^1 + 2 \cdot 3^0$	$[[\emptyset], [\emptyset], \emptyset, \emptyset]$	$\{\{\emptyset\}, \emptyset\}$
9	100	3^2	$[[\emptyset, \emptyset]]$	$\{\{\emptyset\}\}$

A theorem by Euler and some consequences

Theorem (Euler, 1777)

The function $x = f(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ converges when $e^{-e} \leq z \leq e^{1/e}$ and diverges for all other positive z outside this interval.

Remark

If $f(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ converges, then it is a solution of $x = z^x$, or $x^{1/x} = z$.

Corollary

Let Ω be the solution of $\varsigma = \{\varsigma\}$; then its $\mathfrak{A}_\beta$-code, which is a solution of $x = \beta^x$, is defined for every $e^{-e} \leq \beta < 1$ and $1 < \beta \leq e^{1/e}$.

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$\mathfrak{A}_\beta$-code system

Definition

Consider the set system $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$. Given $\beta \in \mathbb{R}^+ \setminus \{1\}$, $e^{-e} \le \beta \le e^{1/e}$, the $\mathfrak{A}_\beta$-*code system* of $\mathscr{S}(\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$ in the real unknowns $x_0, x_1, \ldots, x_n$ is

$$\mathscr{C}_\beta(x_0, x_1, \ldots, x_n) = \begin{cases} x_0 = \beta^{x_{0,1}} + \cdots + \beta^{x_{0,m_0}} \\ x_1 = \beta^{x_{1,1}} + \cdots + \beta^{x_{1,m_1}} \\ \quad\vdots \\ x_n = \beta^{x_{n,1}} + \cdots + \beta^{x_{n,m_n}} \end{cases}$$

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Multiset approximating sequence

- Every h.f. rational hyperset can be approximated by a sequence of h.f. multisets

$$\langle H_i^j \mid 0 \leq i \leq n \rangle = \begin{cases} \langle \emptyset \mid 0 \leq i \leq n \rangle & \text{if } j = 0 \\ \langle [H_{i,1}^j, \ldots, H_{i,m_i}^j] \mid 0 \leq i \leq n \rangle & \text{if } j > 0, \end{cases}$$

- The sequence of their $\mathfrak{A}_\beta$-codes is then meant to approximate the $\mathfrak{A}_\beta$-code of the related hyperset; its $\mathfrak{A}_\beta$-code increment sequence is then defined as

$$\delta_i^j = \mathfrak{A}_\beta(H_i^{j+1}) - \mathfrak{A}_\beta(H_i^j)$$

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Properties of the $\mathfrak{A}_\beta$-code approximating sequence

Lemma

- Each $\mathfrak{A}_\beta$-code approximating value $\mathfrak{A}_\beta(H_i^{j+1})$ is the sum of the $\mathfrak{A}_\beta$-code increment sequence's values until the j-th step.
- The first value of the $\mathfrak{A}_\beta$-code increment sequence is the number of elements of the corresponding set.
- δ_i^{j+1} can be re-written as $\sum_{u=1}^{j} \beta^{\mathfrak{A}_\beta(H_{i,u}^{j})}(\beta^{\delta_i^j} - 1)$.

1. $\beta < 1$.
   - The odd-indexed subsequence of $(\delta_i^j)_{j\in\mathbb{N}}$ is non-positive, while the even-indexed subsequence is non-negative.
   - The sequence $(|\delta_i^j|)_{j\in\mathbb{N}}$ is decreasing.
2. $\beta > 1$.
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A conjecture on $\mathfrak{A}_\beta$

Remark

- The convergence on the $\mathfrak{A}_\beta$-codes of the h.f. well-founded sets and multisets is guaranteed by the convergence of the multiset approximating sequence.
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Consider $\beta \in \mathbb{R}^+$, $e^{-e} \leq \beta < 1$, and $\hbar \in \mathsf{HF}^{1/2}$. Then, there exists and is unique its $\mathfrak{A}_\beta$-code.

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Algebraic and transcendental bases

Remark

Since multisets introduce multiple occurrences of their elements, for every algebraic basis $e^{-e} \leq \beta < 1$ there are issues similar to the one already encountered for $\mathbb{R}_A$: an algebraic basis β satisfies

$$P(\beta) = 0 \quad \text{where} \quad P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \in \mathbb{Z}[x].$$

Therefore, a transcendental basis is preferrable to get injectivity.

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The $\mathfrak{A}_{e^{-1}}$ encoding of h.f. multisets and hypersets is injective over the whole universe $\mathsf{HF}^{1/2} \cup \mathsf{HF}^{\mu}$.

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Open problems

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