On Modal Logic Formulae Minimization

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Introduction

Introduction

In logic, the problem of finding a smaller formula rappresentation historically emerged for propositional logic, due to its applicability to Boolean circuits minimization.

This problem can be naturally generalized to more expressive logics, such as modal logic.

Modal Logic

Modal Logic: Syntax

Given a set of propositional letters $\mathcal{P}$, the set of well-formed formulas of the propositional modal logic ($\mathcal{ML}$) are obtained by the following grammar:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi$$

where the remaining classic Boolean operators can be obtained as shortcuts.

The modality $\Diamond$ (resp., $\Box$) is usually referred to as it is possible that (resp., it is necessary that).

Modal Logic: Syntax

Given a set of propositional letters $\mathcal{P}$, the set of well-formed formulas of the propositional modal logic ($\mathcal{ML}$) are obtained by the following grammar:

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi$$

In the following, we shall use a non-standard, but equivalent grammar to ease both the algorithms and the proof of their properties.

Modal Logic: Semantics

The classical semantics of modal logic is given in terms of Kripke models.

A (finite) Kripke model $K = (W, R, V)$ is composed by a finite set of worlds W, a binary accessibility relation $R \subseteq W \times W$, and a valuation function $V : W \to 2^{\mathcal{P}}$, which associates each world with the set of propositional letters that are true on it.

Modal Logic: Semantics

The satisfiability relation $K, w \models \varphi$ for a generic model K, a generic world $w \in K$, and a formula φ is given by the following clauses:

$$
\begin{array}{lll}
K, w \models p & \text{iff} & p \in V(w), \\
K, w \models \neg\varphi & \text{iff} & K, w \not\models \varphi, \\
K, w \models \varphi \wedge \psi & \text{iff} & K, w \models \varphi \text{ and } K, w \models \psi, \\
K, w \models \varphi \vee \psi & \text{iff} & K, w \models \varphi \text{ or } K, w \models \psi, \\
K, w \models \Diamond\varphi & \text{iff} & \exists v \text{ s.t. } wRv \text{ and } K, v \models \varphi, \\
K, w \models \Box\varphi & \text{iff} & \forall v \text{ s.t. } wRv \text{ it is the case that } K, v \models \varphi.
\end{array}
$$

Moreover, we have

$$
\begin{aligned}
K, w &\models \top, \\
K, w &\not\models \bot.
\end{aligned}
$$

Satisfiability problem

The satisfiability problem for the modal logic is traditionally defined as:

Definition (MSAT)
Given a modal formula φ, does exists a model K, and a world $w \in K$, such that $K, w \models \varphi$?

It is well known that this problem is PSPACE-complete.

In the following we shall call $MSAT()$ any (generic) procedure to solve MSAT.

Formula minimization problems

Measure of size

The minimization problem makes sense when a measure of size is given for the formula φ.

In the following we shall denote with $|\varphi|$ the number of tokens, that is its length. For example:

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In the following we shall denote with $|\varphi|$ the number of tokens, that is its length. For example:

$$\varphi = \Box(p \wedge q) \qquad |\varphi| = 4$$

However, other measures of size can be used.

Formula minimization (Propositional logic)

The classical propositional formula minimization problem is (decisionally) defined as follows:

Definition (PMEF)

Given a propositional formula φ and an integer k, does it exists a formula ψ such that $\psi \equiv \varphi$ and $|\psi| \leq k$?

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This problem is classically defined for formulas in Conjunctive Normal Form (CNF) or Disjunctive Normal Form (DNF), but it can be posed for generic formulas as well. In any of such cases, as it turns out, it is Σ_2^p-complete.

Formula minimization (Propositional logic)

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We shall call PMEF() any (generic) procedure to solve PMEF. More in particular, two classes of approaches exist, namely exact (EXACT-PMEF()) and heuristic (HEURISTIC-PMEF()).

Formula minimization (Modal logic)

Simmetrically to the propositional case, we define the modal minimization problem as:

Definition (MMEF)

Given a modal formula φ and an integer k, does it exists a formula ψ such that $\psi \equiv \varphi$ and $|\psi| \leq k$?

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In this work we propose two algorithms to solve MMEF:

- EXACT-MMEF()
- HEURISTIC-MMEF()

Exact modal minimization

EXACT-MMEF

Theorem

MMEF is *PSPACE*-complete.

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Given a formula φ:

- Enumerate all the candidate formulas, ψ

EXACT-MMEF

Theorem
MMEF is *PSPACE*-complete.

Hardness is easily shown by reduction to satisfiability. As for membership, a naïve approach suffices.

Given a formula φ:

- Enumerate all the candidate formulas, ψ
- Check equivalence via an MSAT checker: $\psi \equiv^? \varphi$.

EXACT-MMEF

Algorithm: Exact Modal Minimal Equivalent Formula

```
function EXACT-MMEF(φ):
    forall k ≤ |φ| do
        forall ψ ∈ Φ(sig(φ)) of length k do
            if EQUIVALENT(φ, ψ) then
                return ψ
            end
        end
    end
    return φ
end
function EQUIVALENT(φ, ψ):
    return not MSAT((φ ∧ ¬ψ) ∨ (ψ ∧ ¬φ))
end
```

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            end
        end
    end
    return φ
end
function EQUIVALENT(φ, ψ):
    return not MSAT((φ ∧ ¬ψ) ∨ (ψ ∧ ¬φ))
end
```

Enumerating all smaller formula is done in a systematic and ordered exploration of the search space.

Such approach guarantees that we do not use more than polynomial space, as it can be implemented without memorizing smaller formulae.

Heuristic modal minimization

HEURISTIC-MMEF

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2. Propositional sub-minimization

HEURISTIC-MMEF

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1. Modal sub-minimization
   a. α threshold for the exact modal minimization
2. Propositional sub-minimization
   b. γ threshold for propositional replacement

HEURISTIC-MMEF

With an heuristic approach we try to avoid the exponential complexity by parametrizing the minimization process:

1. Modal sub-minimization
    a. α threshold for the exact modal minimization
2. Propositional sub-minimization
    b. γ threshold for propositional replacement
    a. β threshold controls exact propositional minimization

HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

```
function HEURISTIC-MMEF($\varphi, \alpha, \beta, \gamma$):
    return HMMEF($\varphi, \alpha, \beta, \gamma, false$)
end
function HMMEF($\varphi, \alpha, \beta, \gamma, isExact$):
    if $|\varphi| \leq \alpha$ then
        return EXACT-MMEF($\varphi$)
    else
        if $\varphi = \psi_1 \circledcirc \psi_2$ and $isExact = false$ then
            $(\overline{\varphi}, \overline{H}) \leftarrow$ PROPOREPLACE($\varphi, \gamma, \emptyset$)
            if $|\overline{\varphi}| \leq \beta$ then
                $(\overline{\varphi}', isExact') \leftarrow$ (EXACT-PMEF($\overline{\varphi}, \gamma$), $true$)
            else
                $(\overline{\varphi}', isExact') \leftarrow$
                  (HEURISTIC-PMEF($\overline{\varphi}, \gamma$), $false$)
            end
            $(\varphi, H) \leftarrow$ PROPOREPLACE($\overline{\varphi}', \gamma, \tilde{H}^{-1}$)
        else
            $isExact' \leftarrow isExact$
        end
        if $\varphi = \psi_1 \circledcirc \psi_2$ then
            return HMMEF($\psi_1, \alpha, \beta, \gamma, isExact'$) $\circledcirc$
              HMMEF($\psi_2, \alpha, \beta, \gamma, isExact'$)
        else if $\varphi = \boxed{\Diamond}\psi$ then
            return $\boxed{\Diamond}$HMMEF($\psi, \alpha, \beta, \gamma, false$)
        end
    end
end
```

HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$**:**
  return *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{false})$
end
function *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{isExact})$**:**
  if $|\varphi| \leq \alpha$ **then**
    return *EXACT-MMEF*(φ)
  else
    if $\varphi = \psi_1 \circledcirc \psi_2$ *and* $\mathit{isExact} = \mathit{false}$ **then**
      $(\overline{\varphi}, \overline{H}) \leftarrow \mathrm{PROPOREPLACE}(\varphi, \gamma, \emptyset)$
      if $|\overline{\varphi}| \leq \beta$ **then**
        $(\overline{\varphi}', \mathit{isExact}') \leftarrow (\mathrm{EXACT\text{-}PMEF}(\overline{\varphi}, \gamma), \mathit{true})$
      else
        $(\overline{\varphi}', \mathit{isExact}') \leftarrow$
         $(\mathrm{HEURISTIC\text{-}PMEF}(\overline{\varphi}, \gamma), \mathit{false})$
      end
      $(\varphi, H) \leftarrow \mathrm{PROPOREPLACE}(\overline{\varphi}', \gamma, \tilde{H}^{-1})$
    else
      $\mathit{isExact}' \leftarrow \mathit{isExact}$
    end
    if $\varphi = \psi_1 \circledcirc \psi_2$ **then**
      return *HMMEF*$(\psi_1, \alpha, \beta, \gamma, \mathit{isExact}') \circledcirc$
       HMMEF$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
    else if $\varphi = \boxdiamond\psi$ **then**
      return $\boxdiamond$*HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
    end
  end
end

modal sub-minimization

propositional sub-minimization

HEURISTIC-MMEF

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$\alpha = 8, \beta = 0, \gamma = 0$

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$$\alpha = 8, \beta = 0, \gamma = 0$$

$$\varphi = (\Box(p \to (r \wedge r))) \wedge ((q \wedge (r \wedge q))) \wedge$$
$$(((\Box q \wedge \Diamond\top) \to \Diamond q) \wedge \Box(p \to (r \wedge r)))$$

HEURISTIC-MMEF

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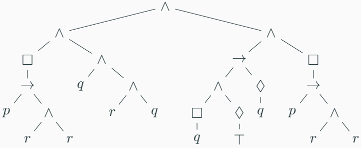

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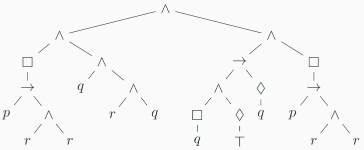

HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

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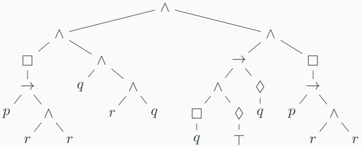

H

key	value
$\Box(p \to (r \wedge r))$	$\tilde{t_1}$
$\Box q$	$\tilde{t_2}$
$\Diamond\top$	$\tilde{t_3}$
$\Diamond q$	$\tilde{t_4}$

HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function $HEURISTIC\text{-}MMEF(\varphi, \alpha, \beta, \gamma)$:
    return $HMMEF(\varphi, \alpha, \beta, \gamma, false)$
end
function $HMMEF(\varphi, \alpha, \beta, \gamma, isExact)$:
    if $|\varphi| \leq \alpha$ **then**
        return $EXACT\text{-}MMEF(\varphi)$
    else
        if $\varphi = \psi_1 \circledcirc \psi_2$ *and* $isExact = false$ **then**
            $(\overline{\varphi}, \overline{H}) \leftarrow \text{PROPOREPLACE}(\varphi, \gamma, \emptyset)$
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            end
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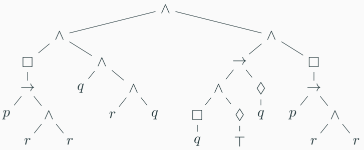
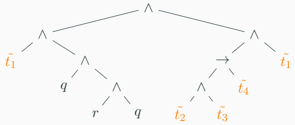
        end
        if $\varphi = \psi_1 \circledcirc \psi_2$ **then**
            return $HMMEF(\psi_1, \alpha, \beta, \gamma, isExact') \circledcirc$
             $HMMEF(\psi_2, \alpha, \beta, \gamma, isExact')$
        else if $\varphi = \boxed{\Diamond}\psi$ **then**
            return $\boxed{\Diamond} HMMEF(\psi, \alpha, \beta, \gamma, false)$
        end
    end
end



H

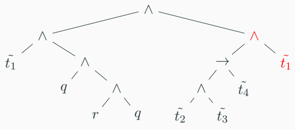
key	value
$\Box(p \to (r \wedge r))$	$\tilde{t_1}$
$\Box q$	$\tilde{t_2}$
$\Diamond\top$	$\tilde{t_3}$
$\Diamond q$	$\tilde{t_4}$



HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$**:**
    return *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{false})$
end
function *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{isExact})$**:**
    if $|\varphi| \leq \alpha$ **then**
        return *EXACT-MMEF*(φ)
    else
        if $\varphi = \psi_1 \circledcirc \psi_2$ *and* $\mathit{isExact} = \mathit{false}$ **then**
            $(\overline{\varphi}, \overline{H}) \leftarrow \text{PROPOREPLACE}(\varphi, \gamma, \emptyset)$
            if $|\overline{\varphi}| \leq \beta$ **then**
                $(\overline{\varphi}', \mathit{isExact}') \leftarrow (\text{EXACT-PMEF}(\overline{\varphi}, \gamma), \mathit{true})$
            else
                $(\overline{\varphi}', \mathit{isExact}') \leftarrow$
                  $(\text{HEURISTIC-PMEF}(\overline{\varphi}, \gamma), \mathit{false})$
            end
            $(\varphi, H) \leftarrow \text{PROPOREPLACE}(\overline{\varphi}', \gamma, \tilde{H}^{-1})$
        else
            $\mathit{isExact}' \leftarrow \mathit{isExact}$
        end
        if $\varphi = \psi_1 \circledcirc \psi_2$ **then**
            return *HMMEF*$(\psi_1, \alpha, \beta, \gamma, \mathit{isExact}') \circledcirc$
              HMMEF$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
        else if $\varphi = \boxed{\Diamond}\psi$ **then**
            return $\boxed{\Diamond}$*HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
        end
    end
end



$\Downarrow \beta = 0$

HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$:
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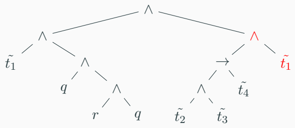
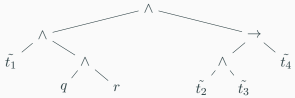
      HMMEF$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
    else if $\varphi = \boxed{\diamond}\psi$ **then**
      return $\boxed{\diamond}$*HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
    end
  end
end



$\Downarrow \beta = 0$



HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$**:**
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      if $|\overline{\varphi}| \leq \beta$ **then**
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      end
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    else
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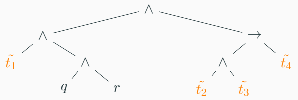
    end
    if $\varphi = \psi_1 \circledcirc \psi_2$ **then**
      return *HMMEF*$(\psi_1, \alpha, \beta, \gamma, \mathit{isExact}') \circledcirc$ *HMMEF*$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
    else if $\varphi = \boxed{\Diamond}\psi$ **then**
      return $\boxed{\Diamond}$*HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
    end
  end
end



HEURISTIC-MMEF

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function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$**:**
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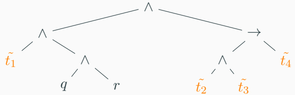
      end
      $(\varphi, H) \leftarrow$ PROPOREPLACE$(\overline{\varphi}', \gamma, \tilde{H}^{-1})$
    else
      $\mathit{isExact}' \leftarrow \mathit{isExact}$
    end
    if $\varphi = \psi_1 \circledcirc \psi_2$ **then**
      return *HMMEF*$(\psi_1, \alpha, \beta, \gamma, \mathit{isExact}') \circledcirc$
      HMMEF$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
    else if $\varphi = \boxed{\Diamond}\psi$ **then**
      return $\boxed{\Diamond}$*HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
    end
  end
end



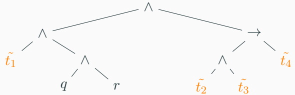
H

key	value
$\Box(p \rightarrow (r \wedge r))$	$\tilde{t_1}$
$\Box q$	$\tilde{t_2}$
$\Diamond \top$	$\tilde{t_3}$
$\Diamond q$	$\tilde{t_4}$

HEURISTIC-MMEF

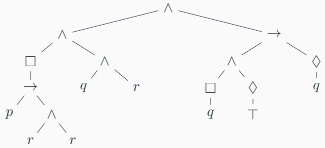
Algorithm: Heuristic Modal Minimal Equivalent Formula

```
function HEURISTIC-MMEF(φ, α, β, γ):
    return HMMEF(φ, α, β, γ, false)
end
function HMMEF(φ, α, β, γ, isExact):
    if |φ| ≤ α then
        return EXACT-MMEF(φ)
    else
        if φ = ψ1 ⊙ ψ2 and isExact = false then
            (φ̄, H̄) ← PROPOREPLACE(φ, γ, ∅)
            if |φ̄| ≤ β then
                (φ̄′, isExact′) ← (EXACT-PMEF(φ̄, γ), true)
            else
                (φ̄′, isExact′) ←
                 (HEURISTIC-PMEF(φ̄, γ), false)
            end
            (φ, H) ← PROPOREPLACE(φ̄′, γ, H̃⁻¹)
        else
            isExact′ ← isExact
        end
        if φ = ψ1 ⊙ ψ2 then
            return HMMEF(ψ1, α, β, γ, isExact′) ⊙
             HMMEF(ψ2, α, β, γ, isExact′)
        else if φ = ◇□ψ then
            return ◇□HMMEF(ψ, α, β, γ, false)
        end
    end
end
```



H

key	value
$\Box(p \to (r \land r))$	$\tilde{t}_1$
$\Box q$	$\tilde{t}_2$
$\Diamond\top$	$\tilde{t}_3$
$\Diamond q$	$\tilde{t}_4$



HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

function *HEURISTIC-MMEF*$(\varphi, \alpha, \beta, \gamma)$**:**
  return *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{false})$
end
function *HMMEF*$(\varphi, \alpha, \beta, \gamma, \mathit{isExact})$**:**
  if $|\varphi| \leq \alpha$ **then**
    return *EXACT-MMEF*(φ)
  else
    if $\varphi = \psi_1 \circledcirc \psi_2$ *and* $\mathit{isExact} = \mathit{false}$ **then**
      $(\overline{\varphi}, \overline{H}) \leftarrow$ PROPOREPLACE$(\varphi, \gamma, \emptyset)$
      if $|\overline{\varphi}| \leq \beta$ **then**
        $(\overline{\varphi}', \mathit{isExact}') \leftarrow$ (EXACT-PMEF$(\overline{\varphi}, \gamma), \mathit{true})$
      else
        $(\overline{\varphi}', \mathit{isExact}') \leftarrow$
        (HEURISTIC-PMEF$(\overline{\varphi}, \gamma), \mathit{false})$
      end
      $(\varphi, H) \leftarrow$ PROPOREPLACE$(\overline{\varphi}', \gamma, \tilde{H}^{-1})$
    else
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    end
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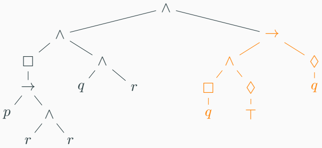
      return *HMMEF*$(\psi_1, \alpha, \beta, \gamma, \mathit{isExact}') \circledcirc$
      HMMEF$(\psi_2, \alpha, \beta, \gamma, \mathit{isExact}')$
    else if $\varphi = \boxdiamond\psi$ **then**
      return $\boxdiamond$ *HMMEF*$(\psi, \alpha, \beta, \gamma, \mathit{false})$
    end
  end
end

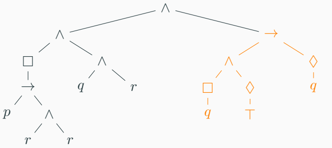


$\Downarrow \alpha = 8$

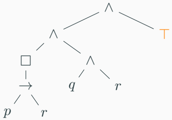
HEURISTIC-MMEF

Algorithm: Heuristic Modal Minimal Equivalent Formula

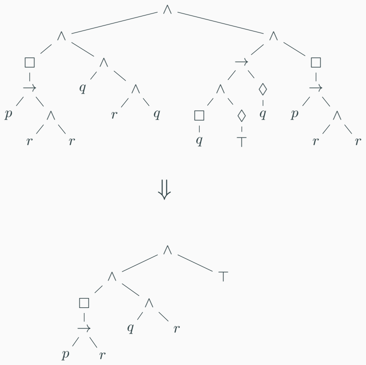
```
function HEURISTIC-MMEF(φ, α, β, γ):
    return HMMEF(φ, α, β, γ, false)
end
function HMMEF(φ, α, β, γ, isExact):
    if |φ| ≤ α then
        return EXACT-MMEF(φ)
    else
        if φ = ψ1 ⊚ ψ2 and isExact = false then
            (φ̄, H̄) ← PROPOREPLACE(φ, γ, ∅)
            if |φ̄| ≤ β then
                (φ̄′, isExact′) ← (EXACT-PMEF(φ̄, γ), true)
            else
                (φ̄′, isExact′) ←
                  (HEURISTIC-PMEF(φ̄, γ), false)
            end
            (φ, H) ← PROPOREPLACE(φ̄′, γ, H̃^−1)
        else
            isExact′ ← isExact
        end
        if φ = ψ1 ⊚ ψ2 then
            return HMMEF(ψ1, α, β, γ, isExact′) ⊚
              HMMEF(ψ2, α, β, γ, isExact′)
        else if φ = ◇□ψ then
            return ◇□HMMEF(ψ, α, β, γ, false)
        end
    end
end
```



$\Downarrow \alpha = 8$



HEURISTIC-MMEF



$$\varphi = (\Box(p \to (r \wedge r))) \wedge ((q \wedge (r \wedge q))) \wedge (((\Box q \wedge \Diamond\top) \to \Diamond q) \wedge \Box(p \to (r \wedge r)))$$

$$\varphi = (\Box(p \to r) \wedge q \wedge r \wedge \top)$$

HEURISTIC-MMEF

Theorem
HEURISTIC-MMEF() is sound, that is, if
$\psi = \text{HEURISTIC-MMEF}(\varphi, \alpha, \beta, \gamma)$, then $\psi \equiv \varphi$ and $|\psi| \leq |\varphi|$.

The heuristic nature of the algorithm does not give us any guarantee on obtaining a minimal formula; however, we can can prove that the algorithm is sound, that is, that we obtain formulas that are equivalent to, and not worse in size than, φ.

HEURISTIC-MMEF

Theorem

HEURISTIC-MMEF() is at least as efficient as EXACT-MMEF()

The complexity of the algorithm strongly depends on its sub-procedure calls. A convenient way to express it is in terms of the parameters α, β, γ.

HEURISTIC-MMEF

Theorem

HEURISTIC-MMEF() is at least as efficient as EXACT-MMEF()

The complexity of the algorithm strongly depends on its sub-procedure calls. A convenient way to express it is in terms of the parameters α, β, γ.

$$\mathcal{C}_{H\text{-}MMEF}(n,\alpha,\beta,\gamma) = O(\frac{n}{\alpha}\mathcal{C}_{E\text{-}MMEF}(\alpha) + n(n2^n\mathcal{C}_{PSAT}(n) + \mathcal{C}_{PMEF}(n) + n)).$$

We observe that in the worst case, we obtain a complexity which is bounded from the top by the one of EXACT-MMEF().

Conclusion

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To summarize, we have:

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- Defined the modal minimization problem

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To summarize, we have:

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- Minimization of formulas in other non-classical logics

Conclusion

To summarize, we have:

- Defined the modal minimization problem
- Developed two algorithms: EXACT and HEURISTIC

Some future steps include:

- Implementation in the Sole.jl learning framework
- Minimization of formulas in other non-classical logics
- Minimization of formulas modulo theory

Thanks for your attention