On two-variable first-order logic with a partial order

Dariusz Marzec, Lidia Tendera

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SAT problem for $\mathcal{FO}^2$

The satisfiability problem for first-order logic, denoted as **Sat($\mathcal{FO}$)** is a decision problem which returns 'yes' iff a first-order sentence φ is satisfiable and 'no' otherwise.

In general, **Sat($\mathcal{FO}$)** is undecidable. However, if we take a sublogic of $\mathcal{FO}$ in which we can use at most two variables in each sentence (denoted as $\mathcal{FO}^2$), we have the following well-known result.

Time complexity for Sat($\mathcal{FO}^2$)

The satisfiability problem for $\mathcal{FO}^2$ is NEXPTIME-complete. (Grädel, Kolaitis,Vardi in 1997)

Expressive power of $\mathcal{FO}^2$

We refer to the set of all predicates for some $\mathcal{FO}$-sentence as a (standard) signature. Note that there are relations which are not expressible in $\mathcal{FO}^2$:

- transitivity: $\forall x \forall y \forall z \big((x < y) \wedge (y < z) \rightarrow (x < z)\big)$,
- linear order: transitivity + antisymmetry + $\forall x \forall y \big((x < y) \vee (y < x)\big)$,
- partial order: transitivity + irreflexivity,
- equivalence: transitivity + symmetry + reflexivity.

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As **Sat($\mathcal{FO}^2$)** is decidable, we often consider satisfiability problems for $\mathcal{FO}^2$ for which the standard signature is extended by distinguished predicates. The most interesting one is transitivity.

Results for **Sat($\mathcal{FO}^2$)** with an extended signature

The table below summarizes known results concerning **Sat** for two-variable first-order logic with a standard signature extended by distinguished relations.

Logic	**Special symbols**	**Number of special symbols**		
		1	2	3 or more
$\mathcal{FO}^2$ NExpTime--complete Grädel, Kolaitis, Vardi (1997)	transitivity	?	undecidable Kieroński (2005), Kazakov (2006)	undecid.
	linear order	NExpTime--complete Otto (2001)	decidable Toruńczyk, Zeume (2022)	undecid. Kieroński (2012)
	equivalence	NExpTime--complete Kieroński, Otto (2005)	2-NExpTime--complete Kieroński, Michaliszyn, Pratt-Hartmann, Tendera (2014)	undecid. Grädel, Otto (1999)

Results for **Sat($\mathcal{FO}^2$)** with an extended signature

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$\mathcal{FO}^2$ with transitivity and its fragments

Let $\sigma = \sigma_0 \cup \{<\}$ where σ_0 is a standard signature and $<$ is a binary relation which is interpreted as a transitive relation. We denote such fragment as $\mathcal{FO}^2$1T. We consider the following sublogics:

- A fragment with a distinguished predicate $<$ is interpreted as a (strict) partial order, here denoted by $\mathcal{FO}^2$1PO. The logic $\mathcal{FO}^2$1T is easily reducible to $\mathcal{FO}^2$1PO.
- The fragment of $\mathcal{FO}^2$1PO where using the negation formal form for some formula, $\exists$-quantifiers can be applied only to conjunctions of the form: $\psi \wedge (x < y) \wedge (y < x)$ or $\psi \wedge \neg(x < y) \wedge (y < x)$ or $\psi \wedge (x < y) \wedge \neg(y < x)$, called the fragment with transitive witnesses, $\mathcal{FO}^2$1PO$_{tw}$.
- The similar fragment of $\mathcal{FO}^2$1PO as above but using conjunctions of the form: $\psi \wedge \neg(x < y) \wedge \neg(y < x)$, called the fragment with free witnesses, $\mathcal{FO}^2$1PO$_{fw}$.

Aim and motivation for our work

- We know that the **finite** satisfiability problem for $\mathcal{FO}^2 1PO$ is decidable (Pratt-Hartmann in 2018).
- We know that **Sat($\mathcal{FO}^2 1PO_{tw}$)** is decidable (Tendera, Szwast in 2019).
- Still, we do not know much about $\mathcal{FO}^2 1PO_{fw}$. First of all, we do not know whether **Sat($\mathcal{FO}^2 1PO_{fw}$)** is decidable. Our paper indicates properties which seem to be critical for solving **Sat($\mathcal{FO}^2 1PO$)** and **Sat($\mathcal{FO}^2 1T$)** .

Aim and motivation for our work

We have also found a restriction of $\mathcal{FO}^2 1\mathrm{PO}_{fw}$ which satisfies the finite model property and has the following features:

- It allows us to express the mutual exclusion property of events in concurrent systems: for two events x and y the formula $\neg\exists x \exists y (Cx \wedge Cy \wedge x \sim y)$ says that x and y cannot access the same critical section at the same time.
- It allows us to express cross product of two classes of elements corresponding to natural statements such as 'elephants are bigger than mice' that are not guarded.

$\mathcal{FO}^2$1PO$_{fw}$ is not locally finite

Result 1

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Notions and symbols

- For a given σ-structure with domain A and a partial order $<$, the interval I of distinct elements $a, b \in A$ is defined as the set of elements 'between' a and b: $I(a, b) = \{b \in A \mid a < c < b\}$.
- We say that a logic is locally finite if for a satisfiable sentence φ in the logic, there is a model such that every interval is finite.

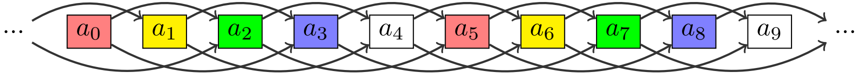
We employ a few abbreviations:

- **x** $\sim$ **y**: $\neg(x < y) \wedge \neg(y < x) \wedge x \neq y$;
- **x** $\bowtie$ **y**: $x < y \vee y < x \vee x = y$;
- **a** mod **b**: $\lfloor a \rfloor_b$.

An axiom of infinity

We present a simple sentence $\Phi_{sp(P)}$ which enforces an infinite model. Let $k = 5$ and let $\{A_i \mid 0 \leq i < k\}$ be standard predicates. We denote to $\Phi_{sp(A)}$ as an *A*-spiral:

$$\exists x\Big(A_0 x\Big) \wedge \wedge_{i=0}^{k-3} \Big(\forall x(A_i x \rightarrow \forall y\big(A_{i+2} y \rightarrow x \bowtie y)\big)\Big) \wedge$$
$$\wedge_{i=0}^{k-1} \Big(\forall x\big(A_i x \rightarrow \exists y(A_{\lfloor i+1 \rfloor_k} y \wedge x \sim y)\big)\Big) \wedge$$
$$\wedge_{i=0}^{k-1} \Big(\forall x\big(A_i x \rightarrow \exists y(A_{\lfloor i-1 \rfloor_k} y \wedge x \sim y)\big)\Big).$$



We show how to 'intertwine' two types of spirals in order to enforce infinite intervals. Let $k = 5$ and let $\{A_i \mid 0 \leq i < k\} \cup \{B_i \mid 0 \leq i < k\}$ be a standard signature. Define Φ:

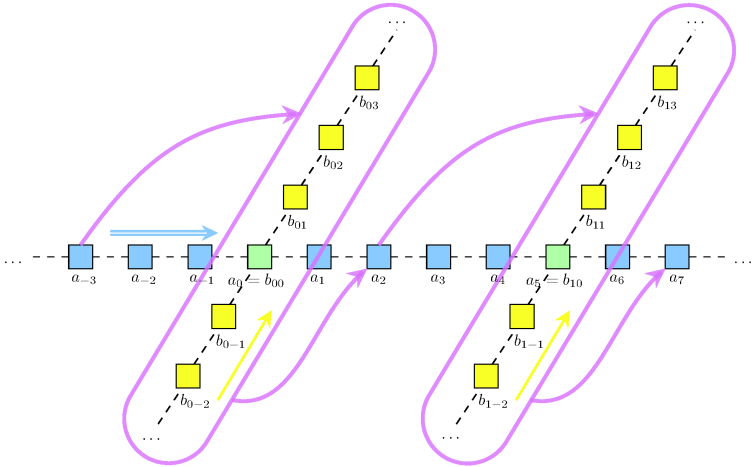
$$\Big(\Phi_{sp(A)} \wedge \Phi_{sp(B)}\Big) \wedge \tag{1}$$

$$\forall x \Big(A_0 x \leftrightarrow B_0 x\Big) \wedge \tag{2}$$

$$\wedge_{i=0}^{k-1} \Big(\forall x (A_2 x \rightarrow \forall y \big(B_i y \rightarrow x \bowtie y)\big)\Big). \tag{3}$$

The sentence Φ is a satisfiable $\mathcal{FO}^2_u 1PO_{fw}$-sentence for which we enforce infinite intervals, so that $\mathcal{FO}^2 1PO_{fw}$ is **not** locally finite.

A model of Φ


. . .
a_{-3}
a_{-2}
a_{-1}
$a_0 = b_{00}$
a_1
a_2
a_3
a_4
$a_5 = b_{10}$
a_6
a_7
b_{01}
b_{02}
b_{03}
b_{0-1}
b_{0-2}
b_{11}
b_{12}
b_{13}
b_{1-1}
b_{1-2}

$\mathcal{F}\mathcal{O}^2$1PO$_{fw}$ has bounded antichain property

Result 2

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Chains and antichains

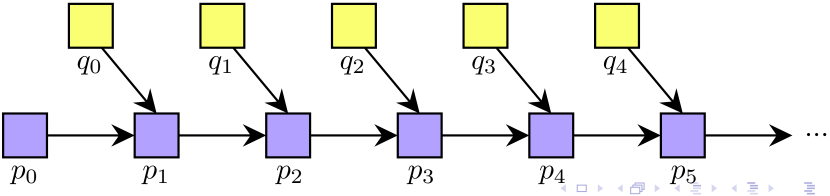
Let $(X, <)$ be a partially ordered set, where $<$ denotes a strict partial order, i.e. a binary relation that is irreflexive and transitive.

- A set $Y \subseteq X$ is a chain if all elements of Y are mutually comparable w.r.t. $<$.
- A set $Y \subseteq X$ is an antichain if all elements of Y are mutually incomparable w.r.t. $<$.
- We say that a logic has bounded antichain property if for a satisfiable sentence φ in the logic, there is a model in which every antichain is finite.

No bounded antichains for $\mathcal{FO}^2$1PO

It turns out that in the general case, the logic $\mathcal{FO}^2$1PO has **no** bounded antichain property. Let $\{P, Q\}$ be a standard signature. The following sentence enforces an infinite antichain:

$$\begin{aligned}\exists x\ Px \wedge \forall x\big(Px \rightarrow \forall y(Py \rightarrow x \bowtie y)\big) \wedge \\ \forall x\big(Qx \rightarrow \forall y(Qy \rightarrow x \sim y)\big) \wedge \\ \forall x\big(Px \rightarrow \exists y(Qy \wedge x \sim y)\big) \wedge \\ \forall x\big(Qx \rightarrow \exists y(Py \wedge x < y)\big).\end{aligned}$$



Bounded antichains for $\mathcal{FO}^2 1PO_{fw}$

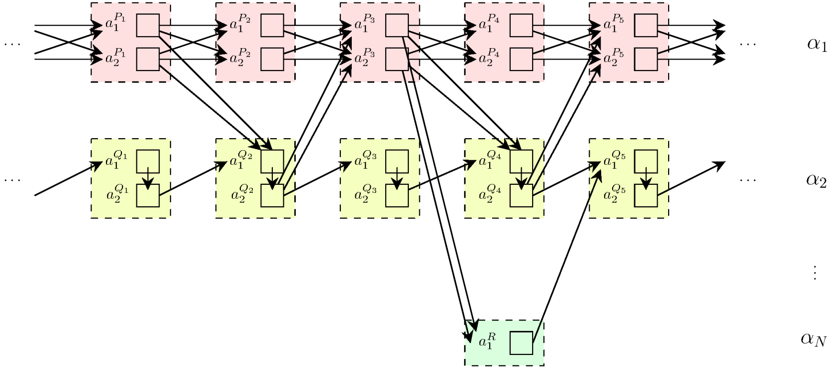
However, if take a sublogic of $\mathcal{FO}^2 1PO$, the fragment $\mathcal{FO}^2 1PO_{fw}$, we are able to show that the logic $\mathcal{FO}^2 1PO_{fw}$ has the bounded antichain property.

Notions:

- Let σ be a signature. We assume that $<$ is included in σ but $=$ is not. A 1-type α is the maximal consistent set of the literals of atomic unary predicates from σ. For a given σ-structure with domain A, we say that $a \in A$ has 1-type α if α is the unique 1-type such that $\alpha[a]$ is true, which is denoted by $\mathrm{tp}[a] = \alpha$.
- Let $P, Q \subseteq A$ be two distinct subsets. The abbreviation $P < Q$ means that for every pair of elements $a \in P$ and $b \in Q$, we have $a < b$.

Factorization for σ-structures

The below σ-structure has a factorization distinguished by dashed rectangles. The elements of the structure have 1-types $\alpha_1, \alpha_2, ..., \alpha_N$.



Factorizations and M-balanced structures

For a given σ-structure with domain A, let a factorization $\mathbb{P}$ be a set of disjoint non-empty subsets of A, called blocks, having following features:

- $\bigcup_{P \in \mathbb{P}} P = A$: the blocks of $\mathbb{P}$ form a partition of A;
- for every block $P \in \mathbb{P}$, there exists a 1-type α such that for every $b \in P$, $\mathrm{tp}[b] = \alpha$: every block has 1-type α;
- for every pair of distinct blocks $P, Q \in \mathbb{P}$, such that all elements in $P \cup Q$ have the same 1-type, we have either $P < Q$ or $Q < P$: blocks with the same 1-type are linearly ordered.

If for every $P \in \mathbb{P}$, the size of the blocks is bounded by a constant M, we say that the σ-structure with a factorization $\mathbb{P}$ is an M-balanced structure.

Maximal factorizations

We can order factorizations w.r.t. $\sqsubseteq$:

- We have a pair of factorizations $\mathbb{P}$ and $\mathbb{Q}$.
- We say that $\mathbb{P} \sqsubseteq \mathbb{Q}$ iff for every block $Q \in \mathbb{Q}$, there exists a block $P \in \mathbb{P}$ such that either $P = Q$ or P can be divided into at least two blocks.
- The relation $\sqsubseteq$ is a (non-strict) partial order.

The following holds due to Zorn's lemma.

Maximal factorization

Every σ-structure has a maximal factorization w.r.t. $\sqsubseteq$.

Basic normal form for unary $\mathcal{FO}^2$1PO$_{fw}$

We consider a fragment of $\mathcal{FO}^2$1PO$_{fw}$ where only unary predicates are allowed in our standard signature, denoted as $\mathcal{FO}^2_u$1PO$_{fw}$. We say that an $\mathcal{FO}^2_u$1PO$_{fw}$-sentence Ψ' is in basic normal form if it is a conjunction of formulas of the following form:

$$\forall x(\alpha(x) \to \forall y(\alpha(y) \to \varphi_\alpha)),$$
$$\forall x(\alpha(x) \to \forall y(\beta(y) \to \varphi_\beta)),$$
$$\forall x(\alpha(x) \to \exists y(\mu(y) \land x \sim y)),$$
$$\forall x\ \mu,\ \exists x\ \mu.$$

We assume that α, β are distinct 1-types, φ_α and φ_β are quantifier-free formulas featuring only $\{<, =\}$ and μ is a quantifier-free unary formula not featuring $<$. The **Sat($\mathcal{FO}^2_u$1PO$_{fw}$)** problem for a sentence φ is easily reducible to the **Sat** problem for a sentence in basic normal form.

Costructing an M-balanced model of φ

Having a satisfiable $\mathcal{FO}^2_u 1\mathrm{PO}_{fw}$-sentence φ in basic normal form and its model with domain A and with maximal factorization $\mathbb{P}$, we can construct an M-balanced model of φ (with $M \geq 2$) as follows:

1. For each $P \in \mathbb{P}$, choose $|P|$ elements from P if $|P| \leq M$ or choose M elements from P otherwise.
2. Crop the model according to the choices from the previous step. We denote the chosen elements in each P by A^P.
3. Set $<$ in a new model as follows:
   - For each block $P \in \mathbb{P}$, set $a \sim a'$ for each distinct $a, a' \in A^P$.
   - For each pair of distinct blocks $P, Q \in \mathbb{P}$ such that neither $P < Q$ nor $Q < P$, set $a \sim a'$ for each $a \in A^P$ and $a' \in A^Q$.
   - For each pair of distinct blocks $P, Q \in \mathbb{P}$ such that $P < Q$, set $a < a'$ for each $a \in A^P$ and $a' \in A^Q$.

Bounded antichain property for $\mathcal{FO}^2 1\mathrm{PO}_{fw}$

We summarize as follows.

M-balanced models for sentences in basic normal form

Let φ be a satisfiable $\mathcal{FO}^2_u 1\mathrm{PO}_{fw}$-sentence in basic normal form and let $M \geq 2$. Then, φ has an M-balanced model.

Soundness of our construction is a consequence of particular features about maximal factorizations.

Extending our proof to the whole logic $\mathcal{FO}^2 1\mathrm{PO}_{fw}$ is not complicated.

Bounded antichain property for $\mathcal{FO}^2 1\mathrm{PO}_{fw}$

Let φ be a satisfiable $\mathcal{FO}^2 1\mathrm{PO}_{fw}$-sentence. Then, φ has a model with bounded antichains.

There is a considerable restriction of $\mathcal{FO}_u^2 1\text{PO}_{fw}$ in basic normal form which has finite model property

Result 3

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Basic normal form for unary $\mathcal{FO}^2 1\mathrm{PO}_{fw}$

Recall that an $\mathcal{FO}^2_u 1\mathrm{PO}_{fw}$-sentence Ψ' is in basic normal form if it is a conjunction of formulas of the following form:

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\end{aligned}$$

We assume that α, β are distinct 1-types, φ_α and φ_β are quantifier-free formulas featuring only $\{<, =\}$ and μ is a quantifier-free unary formula not featuring $<$.

Basic normal form for unary $\mathcal{FO}^2 1\mathrm{PO}_{fw}$

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$$\forall x(\alpha(x) \to \forall y(\beta(y) \to \varphi_\beta)),$$
$$\forall x(\alpha(x) \to \exists y(\mu(y) \wedge x \sim y)),$$
$$\forall x\ \mu,\ \exists x\ \mu.$$

We assume that α, β are distinct 1-types, φ_α and φ_β are quantifier-free formulas featuring only $\{<, =\}$ and μ is a quantifier-free unary formula not featuring $<$.

Our result concerning the unary fragment

It turns out that for an $\mathcal{FO}^2_u 1PO_{fw}$-sentence φ, we can exclude some conjuncts from basic normal form so that we obtain the finite model property.

FMP for a fragment of $\mathcal{FO}^2_u 1PO_{fw}$ in basic normal form

Let φ be a $\mathcal{FO}^2_u 1PO_{fw}$-sentence in basic normal form which does not contain conjuncts of the form:

$$\forall x \Big(\alpha(x) \to \forall y \big(\beta(y) \to x \bowtie y \big) \Big).$$

If φ is satisfiable, then it has a model which size is bounded exponentially in the size of φ.

Future work

- We identified a minimal fragment of $\mathcal{FO}^2 1\mathrm{PO}$ which is critical for answering **Sat($\mathcal{FO}^2 1\mathrm{PO}$)**.
- It is planned to study **Sat($\mathcal{FO}^2 1\mathrm{PO}$)** on scattered structures.
- We believe that the bounded antichain property for $\mathcal{FO}^2 1\mathrm{PO}_{fw}$ allows us to employ some automata techniques to handle scattered structures in the context of $\mathcal{FO}^2 1\mathrm{PO}$.