Decidability of ordered fragments of FOL via modal translation

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Introduction

Ordered fragments of first-order logic (FOL) are fragments in which an ordering associated with the set of variables imposes restrictions on their occurrences in atomic formulas, as well as on the scope of quantifiers.

The simplest of such fragments is due to Quine (1960):

1. an atomic formula can be formed only by giving variables $x_1, ..., x_n$ (in this order) as arguments to an n-ary predicate;
2. a complex formula is either obtained by applying sentential connectives to formulas with the same free variables, or by quantifying over the free variable with the largest index in a formula.

For instance:

$$\forall x_1(Px_1 \to \exists x_2(Rx_1x_2 \wedge \forall x_3 Sx_1x_2x_3)).$$

Introduction

Herzig (1990) provides a translation of Quine's ordered fragment into a propositional modal language in such a way that any input formula is satisfied in a first-order model iff its translation is satisfied in a relational model over a *serial* frame.

Accordingly, one can employ decision procedures for the modal logic **KD** (which is semantically characterized by the class of all serial frames) to decide the satisfiability problem for Quine's ordered fragment.

Introduction

In the present article we provide a simplification and a modification of Herzig's method. By doing so, we obtain decidability and direct model construction for two other ordered fragments of FOL called the *grooved fragment* and the *loosely grooved fragment*, both of which lie between Quine's ordered fragment and the *fluted fragment*.

More precisely, we have the following chain of (strict) inclusions for the expressiveness of the mentioned fragments of FOL:

Quine's $\subset$ *grooved* $\subset$ *loosely grooved* $\subset$ *fluted*

Relational semantics

Language $\mathcal{L}_{MPL}$

Language $\mathcal{L}_{MPL}$ is obtained from the language of Classical Propositional Logic, built over a set $Prop$ of atomic propositions, by adding the modal operator $\Box$ ('it is necessary that').
The operator $\Diamond$ ('it is possible that') can be defined as $\neg\Box\neg$.

Frame

A *frame* to interpret $\mathcal{L}_{MPL}$ is a pair $\mathfrak{F} = \langle W, R \rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$ is an accessibility relation.

Model

A model over a frame $\mathfrak{F} = \langle W, R \rangle$ is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where $V : Prop \to \wp(W)$ is called *valuation*.

Relational semantics

Truth-conditions

Let $\mathfrak{M} = \langle W, R, V \rangle$ be any model and $w \in W$.

- $\mathfrak{M}, w \vDash p \iff w \in V(p)$
- $\mathfrak{M}, w \vDash \neg\varphi \iff \mathfrak{M}, w \nvDash \varphi$
- $\mathfrak{M}, w \vDash \varphi \vee \psi \iff \mathfrak{M}, w \vDash \varphi$ or $\mathfrak{M}, w \vDash \psi$
- $\mathfrak{M}, w \vDash \Box\varphi \iff (\forall v \in W)\ Rwv$ implies $\mathfrak{M}, v \vDash \varphi$

Relational semantics

Satisfiability in models

A formula φ is *satisfiable in a model* $\mathfrak{M}$ iff, for some $w \in W$, we have $\mathfrak{M}, w \models \varphi$.

Validity in models

A formula φ is *valid in a model* $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) iff, for all $w \in W$, we have $\mathfrak{M}, w \models \varphi$.

Validity in frames

A formula φ is *valid in a frame* $\mathfrak{F}$ (notation: $\mathfrak{F} \models \varphi$) iff, for all $w \in W$ and all models $\mathfrak{M}$ over $\mathfrak{F}$, we have $\mathfrak{M}, w \models \varphi$.

Propositional modal logics

K-validity and **K**-satisfiability

A formula φ is **K**-valid iff it is valid in all frames; it is **K**-satisfiable iff it is satisfied in a model.

Serial frames

A frame $\langle W, R \rangle$ is serial iff $\forall w \exists v Rwv$.

KD-validity and **KD**-satisfiability

A formula φ is **KD**-valid iff it is valid in all serial frames; it is **KD**-satisfiable iff it is satisfied in a model over a serial frame.

(Strong) finite model property

Filtration

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model, φ be a formula, and Σ be the set of subformulas of φ.

For all $\psi \in \Sigma$ and $w \in W$, we have $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}^f_\Sigma, w^f_\Sigma, \models \psi$, where $\mathfrak{M}^f_\Sigma$ is the model obtained by performing filtration f through Σ on model $\mathfrak{M}$.

Also, $|\mathfrak{M}^f_\Sigma| \leq 2^{|\Sigma|}$.

Filtrations preserve seriality: if $\mathfrak{M}$ is a model over a serial frame, so is $\mathfrak{M}^f_\Sigma$.

Bounded morphism

Bounded morphism

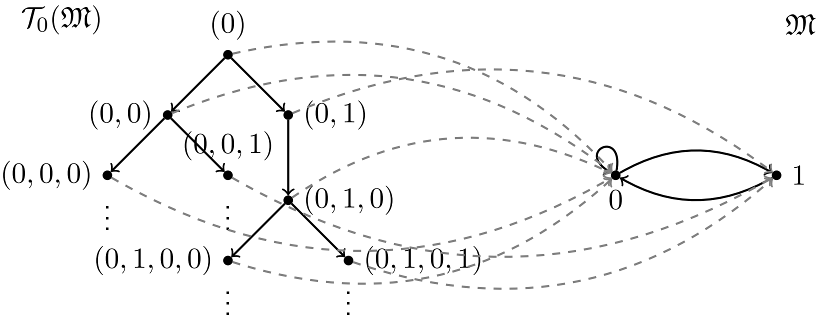
Let $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ be two models. A function $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ is a *bounded morphism* if satisfies:

- $w \in V(p)$ iff $f(w) \in V'(p)$, for $p \in Prop$,
- if Rwv, then $R'f(w)f(v)$, i.e. f is a homomorphism with respect to R,
- if $R'f(w)v'$, then $\exists v \in W$ s.t. $f(v) = v'$ and Rwv.

Invariance

Let $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ be two models s.t. $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ is a bounded morphism. Then for each formula φ and $w \in W$ we have $\mathfrak{M}, w \vDash \varphi$ iff $\mathfrak{M}', f(w) \vDash \varphi$.

Tree model property



The dashed arrows represent a bounded morphism.

Quine's ordered fragment

Ordered formulas

The set of ordered formulas is defined inductively as follows:

1. For any n-ary predicate P, $Px_1 \ldots x_n$ is an ordered formula of level n.
2. If ϕ and ψ are ordered formulas of level n, so are $\neg\phi$ and $(\phi \wedge \psi)$.
3. If ϕ is an ordered formula of level n $(n > 0)$, then $\forall x_n \phi$ is an ordered formula of level $n - 1$.

Note that an ordered formula of level 0 is a *sentence*.

Quine's ordered fragment

Example

- $\forall x_1(Px_1 \rightarrow \exists x_2(Rx_1x_2 \wedge \forall x_3 Sx_1x_2x_3))$

Non-examples

- $\forall x_1(Px_1 \rightarrow \exists x_2(Rx_2x_2 \wedge \forall x_3 Sx_1x_2x_3))$
- $\forall x_1(Px_1 \rightarrow \exists x_2 \forall x_3(Rx_2x_3 \wedge Cx_1x_2x_3))$
- $\forall x_1(Px_1 \rightarrow \forall x_2(Rx_2x_1 \wedge \exists x_3 Cx_1x_2x_3))$

First-order semantics

Satisfaction

Let $\mathcal{M} = \langle D, I \rangle$ be a model. Let $D^* = \bigcup_{n \in \mathbb{N}} D^n$. We write σ_n for an n-tuple in D^*, and ϕ_n, ψ_n for ordered formulas of level n.
Then:

- $\mathcal{M}, \sigma_n \vDash P x_1 \ldots x_n$ iff $\sigma_n \in I(P)$
- $\mathcal{M}, \sigma_n \vDash \neg \phi_n$ iff it is not the case that $\mathcal{M}, \sigma_n \vDash \phi_n$
- $\mathcal{M}, \sigma_n \vDash \phi_n \wedge \psi_n$ iff $\mathcal{M}, \sigma_n \vDash \phi_n$ and $\mathcal{M}, \sigma_n \vDash \psi_n$
- $\mathcal{M}, \sigma_{n-1} \vDash \forall x_n \phi_n$ iff for all $a \in D$, $\mathcal{M}, \sigma_{n-1} a \vDash \phi_n$

In particular, for an ordered sentence ϕ and the empty tuple ϵ, we follow the usual practice and write $\mathcal{M} \vDash \phi$.

Translation (Herzig 1990)

Translation function

The translation function, tr, from ordered formulas to modal formulas is defined recursively as follows:

- $tr(P_i x_1 \dots x_n) = p_i$
- $tr(\neg\phi) = \neg tr(\phi)$
- $tr(\phi \wedge \psi) = tr(\phi) \wedge tr(\psi)$
- $tr(\forall x \phi) = \Box tr(\phi)$

Adequacy of the translation

Let $\mathcal{M} = \langle D, I \rangle$ be a first-order model. We then define the Kripke model $\mathfrak{M} = \langle W, R, V \rangle$ as follows:

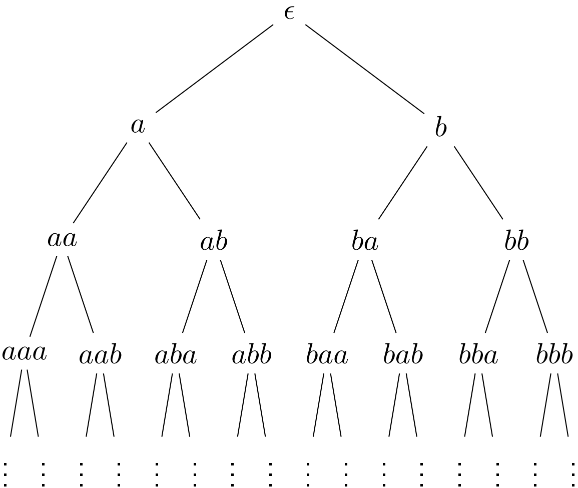
- $W = D^* = \bigcup_{n \in \mathbb{N}} D^n$
- for any $\sigma, \tau \in W$, $R\sigma\tau$ iff $\tau = \sigma a$ for some $a \in D$
- $V(p_i) = I(P_i)$

Lemma 1

Let $\mathcal{M} = \langle D, I \rangle$ and $\mathfrak{M} = \langle W, R, V \rangle$ be the models defined above. For $n \geq 0$, if ϕ_n is an ordered formula of level n and σ_n is an n-tuple in D^*, then: $\mathcal{M}, \sigma_n \vDash \phi_n$ iff $\mathfrak{M}, \sigma_n \vDash tr(\phi_n)$.

Adequacy of the translation

$D = \{a, b\}$



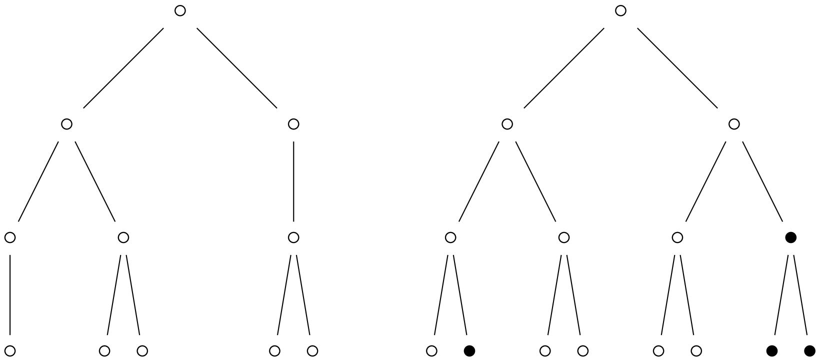
Adequacy of the translation

For the other direction, we can assume, without loss of generality, that $tr(\phi)$ is satisfied at the root of a tree model where the tree is finitely branching.

Lemma 2

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model where $\langle W, R \rangle$ is a serial m-ary tree ($m > 0$) and $w_0 \in W$ its root. Then there is a model $\mathfrak{M}' = \langle W', R', V' \rangle$, where $\langle W', R' \rangle$ is a perfect m-ary tree and $w'_0 \in W'$ its root, s.t. there is a surjective bounded morphism $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ and, in particular, $f(w'_0) = w_0$.

Adequacy of the translation



Decidability of Quine's ordered fragment

Theorem 1
Let ϕ be an ordered sentence. Then: ϕ is satisfiable iff $tr(\phi)$ is **KD**-satisfiable.

Therefore, the satisfiability problem for the ordered fragment is decidable.

The grooved fragment

Grooved formulas

The set of grooved formulas is the smallest set satisfying the following conditions:

1. For each n-ary predicate P $(n > 1)$, $Px_1 \ldots x_n$ is a grooved formula of level n.
2. For each unary predicate P, Px_n is a grooved formula of level n $(n > 0)$.
3. If ϕ and ψ are grooved formulas of level n, so are $\neg\phi$ and $(\phi \wedge \psi)$.
4. If ϕ is a grooved formula of level n $(n > 0)$, then $\forall x_n \phi$ is a grooved formula of level $n - 1$.

The grooved fragment

The identification of the grooved fragment is inspired by works on the *relational syllogistic*, the extension of the classical syllogistic with relational terms.

- No student admires every professor
  $\forall x_1(Sx_1 \to \neg\forall x_2(Px_2 \to Ax_1x_2))$
- No lecturer introduces any professor to every student
  $\forall x_1(Lx_1 \to \neg\exists x_2(Px_2 \wedge \forall x_3(Sx_3 \to Ix_1x_2x_3)))$

Clearly, such sentences are not ordered, since they typically contain atoms of the form Px_n, where $n > 1$, whereas this is now accommodated by the grooved fragment.

The grooved fragment

Satisfaction

Let $\mathcal{M} = \langle D, I \rangle$ be a first-order model. We write σ_n for an n-tuple from D^*, and ϕ_n, ψ_n for grooved formulas of level n. Then:

- $\mathcal{M}, \sigma_n \vDash Px_1 \ldots x_n$ iff $\sigma_n \in I(P)$ (P is not unary)
- $\mathcal{M}, \sigma_{n-1}a \vDash Px_n$ iff $a \in I(P)$ (P is unary)
- $\mathcal{M}, \sigma_n \vDash \neg\phi_n$ iff it is not the case that $\mathcal{M}, \sigma_n \vDash \phi_n$
- $\mathcal{M}, \sigma_n \vDash \phi_n \wedge \psi_n$ iff $\mathcal{M}, \sigma_n \vDash \phi_n$ and $\mathcal{M}, \sigma_n \vDash \psi_n$
- $\mathcal{M}, \sigma_{n-1} \vDash \forall x_n \phi_n$ iff for all $a \in D$, $\mathcal{M}, \sigma_{n-1}a \vDash \phi_n$

Translation

Translation function

The translation function, tr, from frooved formulas to modal formulas is defined recursively as follows:

- $tr(Px_1 \ldots x_n) = p$ (P is not unary)
- $tr(Px_n) = p$ (P is unary)
- $tr(\neg\phi) = \neg tr(\phi)$
- $tr(\phi \wedge \psi) = tr(\phi) \wedge tr(\psi)$
- $tr(\forall x \phi) = \Box tr(\phi)$

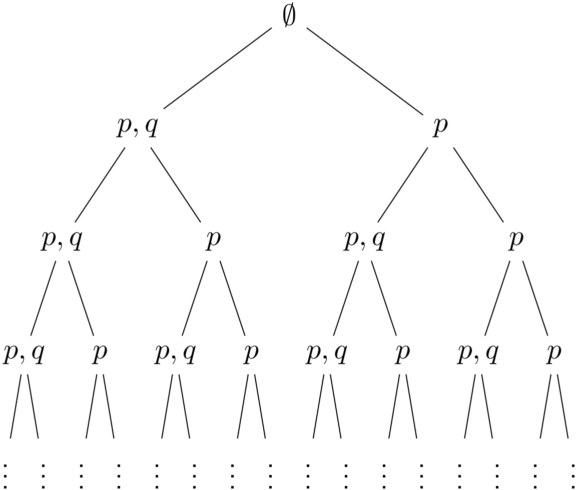
Adequacy of the translation

Let $\mathcal{M} = \langle D, I \rangle$ be a model for $\mathcal{L}_{FOL}$. Then the $\mathcal{L}_{PML}$-*analogue* of $\mathcal{M}$, $\mathfrak{M} = \langle W, R, V \rangle$, is a model for $\mathcal{L}_{PML}$ defined as below:

- $W = D^*$
- for any $\sigma, \tau \in W$, $R\sigma\tau$ iff $\tau = \sigma a$ for some $a \in D$
- for non-unary P_i, $V(p_i) = I(P_i)$
- for unary P_i, $V(p_i) = \{\sigma \in D^* \setminus \{\epsilon\} : last(\sigma) \in I(P_i)\}$

$\mathfrak{M}$ is obviously a **KD**-model over a tree.

Adequacy of the translation



Adequacy of the translation

Given a grooved sentence ϕ, let $\mathbf{S}(\phi)$ be the set of propositional variables corresponding to the *unary* predicates in ϕ. Let $\Gamma(\phi)$ be the set of maximal consistent sets of literals (i.e. propositional variables or their negation) formed by elements of $\mathbf{S}(\phi)$, and let

$$\Psi(\phi) = \left\{ \bigwedge \Sigma : \Sigma \in \Gamma(\phi) \right\}$$

$$\Upsilon(\phi) = \left(\bigwedge_{\psi \in \Psi(\phi)} (\Diamond\psi \to \Box\Diamond\psi) \right) \wedge \left(\bigwedge_{\psi \in \Psi(\phi)} (\neg\Diamond\psi \to \Box\neg\Diamond\psi) \right)$$

Lemma 3

Let ϕ be a grooved sentence. If ϕ is satisfiable, then $tr(\phi)$ is satisfied in a **KD**-model where $\Upsilon(\phi)$ is globally true.

Adequacy of the translation

For a **KD** model in which $\Upsilon(\phi)$ is globally true, a tree-unraveling is a *well-sorted* tree model.

A well-sorted tree model can be expanded into a *fulfilled* tree model, where the frame is a perfect tree.

Lemma 4

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a well-sorted tree model over a serial m-ary tree $(m > 0)$ with root w_0. Then there is a fulfilled tree model $\mathfrak{M}' = \langle W', R', V' \rangle$ over a perfect m'-ary tree $(m' \geq m)$ with root w_0', and a surjective bounded morphism $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ s.t. $f(w_0') = w_0$.

Decidability of the grooved fragment

Theorem 2

Let ϕ be a grooved sentence. ϕ is satisfiable in first-order logic iff $tr(\phi)$ is satisfied in a **KD**-model in which $\Upsilon(\phi)$ is globally true,

Therefore, the satisfiability problem for the grooved fragment is decidable.

In the paper we also defined the loosely grooved fragment, which is more expressible then the grooved one. Each loosely grooved sentence can be rewritten into a satisfiability-equivalent grooved sentence.

References

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- A. Herzig (1990). A new decidable fragment of first-order logic, *Abstracts of the 3rd Logical Biennial Summer School and Conference in Honour of S. C. Kleene*.